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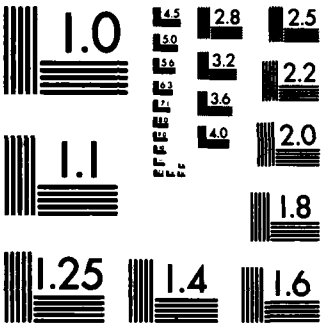

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AD A131849

An Alternative to the Monte Carlo

Estimation of Network Reliability

George S. Fishman

Technical Report No. UNC/ORSA/TR-83/2
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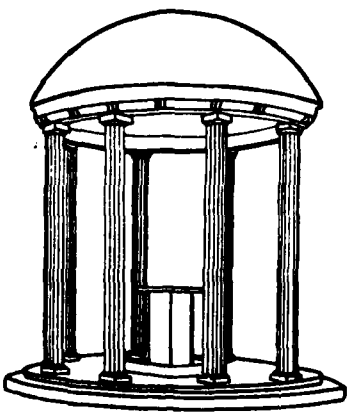
AT CHAPEL HILL



An Alternative to the Monte Carlo Estimation of Network Reliability

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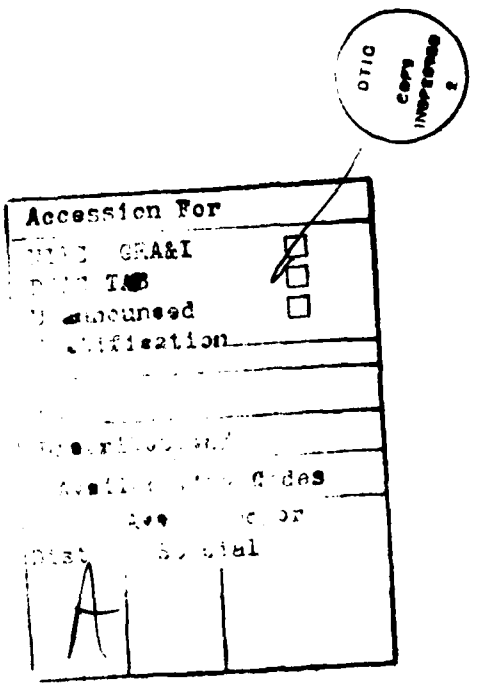
Curriculum in Operations Research
and Systems Analysis
University of North Carolina at Chapel Hill

This research was supported, in part, by the Office of Naval Research under Contract N00014-76-C-0302. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Acknowledgment

I wish to express my sincere appreciation to Professor J. Scott Provan for many helpful conversations on the problem of network reliability.



Abstract

This paper describes a numerical method based on quasirandom points for estimating the probability $g(s,t)$ that vertices s and t are connected in an undirected network $G=(V,M)$ with $|V|$ perfect vertices and $|M|$ randomly and independently failing edges. It is shown that the error of estimate, as measured in terms of extreme discrepancy, has a lower bound proportional to $(\log K)^{|P(s,t)|/2}/K$ for all possible sampling sequences, but has an upper bound proportional to $(\log K)^{|P(s,t)|}/K$ for certain quasirandom sequences, where $P(s,t) \subseteq M$ is the union of all minimal pathsets between s and t and K is the number of replications. By comparison, previously proposed sampling methods for this problem all lead to a standard error of estimate proportional to $K^{-1/2}$. Moreover, since quasirandom points are not random, the associated bounds are deterministic. By using a minimal cutset $R(s,t)$ with a certain special property these lower and upper bounds become, respectively, $(\log K)^{|P(s,t)-R(s,t)|/2}/K$ and $(\log K)^{|P(s,t)-R(s,t)|}/K$. This suggests that one choose a cutset $R(s,t)$ with maximal cardinality. Also, it is shown that the coefficient of $(\log K)^{|P(s,t)-R(s,t)|}/K$ is least if among all cutsets with maximal cardinality one chooses the cutset with the largest failure probability. The paper extends the results to more global measures of reliability and to conditional measures of reliability. A discussion of the computation time complexity of the proposed method is also included.

Key Words and Phrases

Monte Carlo method

Network reliability

Quasirandom points

Variance reduction

Introduction

Consider an undirected network $G = (V,M)$ of $|V|$ vertices and $|M|$ edges where the vertices always operate and where edge i has probability q_i of operating $i \in M$. Also assume that edges fail independently. Suppose that one wants to compute $g = g(s,t)$, the probability that s and t $s,t \in V$ are connected. Let $x_i = 1$ if edge i operates and $x_i = 0$ if edge i fails. Let $\phi(x_1,\ldots,x_{|M|}; s,t)=1$ if the vector $(x_1,\ldots,x_{|M|})$ admits at least one operating path between s and t and $\phi(x_1,\ldots,x_{|M|}; s,t) = 0$ otherwise. Then

$$g(s,t) = \sum_{x_1=0}^{1} \cdots \sum_{x_{|M|}=0}^{1} \phi(x_1,\ldots,x_{|M|}\ ; s,t) \prod_{i=1}^{|M|} q_i^{x_i}(1-q_i)^{1-x_i} \qquad (1)$$

$$s,t \in V \ .$$

Since the computation time complexity of (1) is $O(2^{|M|})$, direct computation can only be performed for relatively small $|M|$. To overcome the computational difficulty, at least three different approaches have been proposed. The first concentrates on networks with special structure. For example, Rosenthal (1977) describes a time-saving decomposition method that applies to loosely connected networks. The effect of this approach is to make the time complexity $O(2^J)$ where $0 < J < |M|$. The second concentrates on finding bounding inequalities for the reliability measure at a cost that grows polynomially in $|M|$. For example, see Zemel (1980).

The third approach uses the Monte Carlo method and, in particular, stratified sampling techniques. See van Slyke and Frank (1972) and Diegert and Diegert (1981). In principle, the Monte Carlo method allows an investigator to make a tradeoff between the cost of computation and the accuracy of computation. Although all methods of evaluation on computers lead to some numerical roundoff error, presumably one adopts

methods of calculation that keep this error within tolerable bounds. To this roundoff error the Monte Carlo method adds a sampling error incurred by performing less than a total enumeration of all the possible states for $(x_1,\ldots,x_{|M|})$ when estimating the reliability measure g. For K replications using independent random sampling, one estimates g with standard error proportional to $1/K^{1/2}$ at a cost $O(K)$ for a given graph G. For an experimental design that employs pure random sampling, this error is $\sqrt{g(1-g)/K}$. Provided that the technique is judiciously used, the effect of stratified sampling is to reduce the numerator of this quantity. But convergence with regard to K remains proportional to $1/K^{1/2}$.

The purpose of this paper is to describe how one can employ a numerical method based on quasirandom points to accelerate the rate of convergence of the error of the estimate of g. Quasirandom points is a specialized topic related to the Monte Carlo method whose value for problem solution is most evident in a multivariable setting. In this paper the operating statuses of the edges constitute this setting. Niederreiter (1978) contains a comprehensive survey of the theory of quasirandom points and Schmidt (1977) provides rigorous proofs of all theorems related to quasirandom points used in the present paper.

After introducing the relevant nomenclature, Section 1 describes in detail the effect of pure random sampling in a Monte Carlo experiment on the error of the estimate. It then describes the effect on this error of replacing pure random sampling by an experimental design based on quasirandom points. In particular, when estimating $g(s,t)$, the probability that nodes s and t are connected, it shows that, for given G, errors have a deterministic lower bound proportional to $(\log K)^{|P(s,t)|/2}/K$ for

all possible sampling sequences, but have a deterministic upper bound proportional to $(\log K)^{|P(s,t)|}/K$ for certain quasirandom sequences, where $P(s,t)$ is the union of all pathsets between s and t.† Section 2 shows how further improvement in the rates of convergence of the error is possible by using information available about cutsets of the network. In particular, it is shown that among all cutsets for which the paths from s to t each contain exactly one edge in a cutset, using the cutset $R(s,t)$ with maximal cardinality leads to the most rapidly converging bound on the error of estimate when quasirandom points are employed. Most notable is the replacement of $(\log K)^{|P(s,t)|/2}$ and $(\log K)^{|P(s,t)|}$ by $(\log K)^{|P(s,t)-R(s,t)|/2}$ and $(\log K)^{|P(s,t)-R(s,t)|}$, respectively, in the bounds on convergence. Also, among all such cutsets, choosing the one with the largest failure probability gives the smallest coefficient on $(\log K)^{|P(s,t)-R(s,t)|}/K$ in the upper bound.

Section 3 extends the analysis to the estimation of other reliability measures, some more global than $g(s,t)$ and others involving conditionality. The global measures include $g(s,T)$, the probability that vertex s is connected to all vertices $t \in T$ and $h(s,T)$, the mean number of s-t connections $t \in T$. The conditional measures include $g(s,t|v)$, the probability that s and t are connected given that edge v fails, and $g(v|s,t)$, the probability that edge v fails given that s and t are not connected. The first of these characterizes the criticality of v whereas the second provides information about a potential source of the failure to communicate between s and t.

Section 4 discusses computation time complexity for the proposed method when using quasirandom points alone as in Section 1 and when combined with a cutset as in Section 2.

†All pathsets and cutsets are assumed minimal.

1. Quasirandom Points

We begin by introducing a relatively general setting for assessing error in a multivariable Monte Carlo experiment. Let I_N denote the N-dimensional unit hypercube $[0,1]^N$, let $W_N = \pi_{i=1}^N [0,\beta_i)$ with $0 \le \beta_i < 1$ $i=1,\ldots,N$ denote an arbitrary box in I_N and let $u = \{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \ldots\}$ denote a sequence of points in I_N. Also, let

$$\lambda(W_N) = \pi_{i=1}^N \beta_i \tag{1a}$$

and

$$A(W_N;K) = \text{number of points } \underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_K \text{ that are in } W_N . \tag{1b}$$

Then

$$\Delta(W_N;K) = |K^{-1}A(W_N;K) - \lambda(W_N)| \tag{1c}$$

measures the absolute error or discrepancy incurred in approximating $\lambda(W_N)$ by $K^{-1}A(W_N;K)$. For a particular sequence u, one has the extreme discrepancy

$$D_K = D(N,K,u) = \sup_{W_N \subset I_N} \Delta(W_N;K) \tag{2a}$$

and the L^2 discrepancy

$$T_K = T(N,K,u) = [\int_0^1 \cdots \int_0^1 \Delta^2(W_N;K) d\beta_1,\ldots,d\beta_N]^{1/2} . \tag{2b}$$

Whereas D_K leads to a worst case assessment, T_K leads to an averaging of mean-square errors. Note that $T_K \le D_K$.

With regard to the graph G, let

$$L(s,t) = \text{set of pathsets between } s \text{ and } t \tag{3a}$$

$$P_r(s,t) = \text{pathset } r \text{ between } s \text{ and } t, \; r \in L(s,t) \tag{3b}$$

$$W(P_r(s,t)) = \pi_{i \in P_r(s,t)} [0,q_i) \tag{3c}$$

and

$$B_j(P_r(s,t)) = \pi_{i \in P_r(s,t)} I_{[0,q_i)}(u_{ij}) \tag{4}$$

where

$$I_{[a,b)}(x) = 1 \quad \text{if} \quad a \le x < b$$

$$= 0 \quad \text{otherwise} ,$$

$0 \le u_{ij} < 1$ and $\underset{\sim}{u}_j=(u_{ij};\ i\epsilon\cup_{r\epsilon L(s,t)}P_r(s,t))$ $j=1,2,\ldots$.
Here the subscript j denotes replication number. As we show shortly, the selection of a generator for $\{u_{ij}\}$ determines the nature of the error incurred in estimating g by the Monte Carlo method. For convenience of exposition we suppress the (s,t) arguments unless they are needed to avoid ambiguity.

Observe that

$$B_j(\cup_{i=1}^{m} P_{r_i}) = \pi_{i=1}^{m} B_j(P_{r_i}) \qquad r_1,\ldots,r_m\epsilon L . \tag{5}$$

Finally, we make repeated use of the expansion

$$\pi_{i=1}^{n} (1-y_i) = 1 + \sum_{k=1}^{n}(-1)^k \sum_{1\le m_1<\ldots<m_k\le n} \pi_{r=1}^{k} y_{m_r} . \tag{6}$$

Note that if $B_j(P_r)=1$ for some $r\epsilon L$ then s and t are connected on replication j. If $\pi_{r\epsilon L}[1-B_j(P_r)]=1$ then s and t are not connected on replication j. Then regardless of the sampling plan employed the probability that s and t are connected on replication j is

$$g_j(s,t) = pr[\cup_{r\epsilon L}(B_j(P_r)=1)] \tag{7}$$

and the <u>conditional</u> probability that s and t are connected is

$$\begin{aligned} g(s,t)|\underset{\sim}{u}_j) &= 1-\pi_{r\epsilon L}[1-B_j(P_r)] \\ &= \sum_{m=1}^{|L|}(-1)^{m+1} \sum_{r_1<\ldots<r_m\epsilon L} B_j(\cup_{i=1}^{m} P_{r_i}) . \end{aligned} \tag{8}$$

As a first estimator of g we consider

$$\hat{g}_K = \hat{g}_K(s,t) = \frac{1}{K} \sum_{j=1}^{K} g(s,t|\underset{\sim}{u}_j) \ . \tag{9a}$$

From (1), (4) and (5) this estimator can be written equivalently as

$$\hat{g}_K = \frac{1}{K} \sum_{m=1}^{|L|} (-1)^{m+1} \sum_{r_1<\ldots<r_m \epsilon L} A(\cup_{i=1}^{m} W(P_{r_i}); K) \tag{9b}$$

so that

$$\hat{g}_K - g = \sum_{m=1}^{|L|} (-1)^{m+1} \sum_{r_1<\ldots<r_m \epsilon L} \Delta(\cup_{i=1}^{m} W(P_{r_i}); K) \ . \tag{10a}$$

From (2a) one has the extreme discrepancy bound

$$|\hat{g}_K - g| \le D^*(G,K,u) = \sum_{m=1}^{|L|} \sum_{r_1<\ldots<r_m \epsilon L} D(|\cup_{i=1}^{m} P_{r_i}|,K,u) \tag{10b}$$

and from (3) the L^2 discrepancy bound

$$\begin{aligned} &\left[\int_0^1 \ldots \int_0^1 |g_K - g|^2 \, dq_1 \ldots dq_{|M|}\right]^{1/2} \le T^*(G,K,u) \\ &\qquad = \sum_{m=1}^{|L|} \sum_{r_1<\ldots<r_m \epsilon L} T(|\cup_{i=1}^{m} P_{r_i}|,K,u) \ . \end{aligned} \tag{10c}$$

Observe that $P=\cup_{r \epsilon L} P_r$ is the set of all arcs in the pathset L. If u_{ij} $i \epsilon P$ $j=1,\ldots,K$ in (4) were independent random variables drawn from the uniform distribution on $[0,1)$, then $\hat{g}_K$ would be unbiased with $\text{var}\ \hat{g}_K = g(1-g)/K$. Since $(\text{var}\ \hat{g}_K)^{1/2}$, the <u>standard error</u> of $\hat{g}_K$, converges as $K^{-1/2}$, $D^*(G,K,u)$ cannot converge faster than $K^{-1/2}$ for pure random sampling. In fact, for N dimensions

$$\limsup_{K \to \infty} \frac{\sqrt{2K}\, D(N,K,u)}{\sqrt{\log \log K}} = 1 \qquad \text{w.p. 1,} \tag{11}$$

a result due to Chung (1949) for $N=1$ and to Kiefer (1961) for $N > 1$.

Also, Hammersley (1960) shows that for pure random sampling

$$ET_K^2 = (2^{-N} - 3^{-N})/K \quad .$$

How Well Can a Numerical Method Perform?

We now turn to an assessment of how well one can expect any numerical method to perform with regard to convergence as a function of K using the error bounds $D^*(N,K,u)$ and $T^*(N,K,u)$. Clearly, one would like to choose the vector sequence u so that convergence is as rapid as possible. The theorems 1 through 3 provide some useful insights.

Theorem 1 (van Aardanne-Ehrenfest 1945). For any infinite sequence u with $N \geq 1$

$$\lim_{K\to\infty} KD(N,K,u) = \infty \quad . \tag{12}$$

Theorem 2 (Roth 1954). For any sequence u of K points with $N \geq 2$

$$T(N,K,u) \geq G_N K^{-1} (\log_2 K)^{(N-1)/2} \tag{13}$$

where

$$G_N = 2^{-4N}(N-1)^{(1-N)/2} \quad .$$

Theorem 3 (Roth 1954). For any infinite sequence u with $N \geq 1$

$$T(N,K,u) \geq G_N K^{-1} (\log_2 K)^{N/2} \quad . \tag{14}$$

Also, Roth (1980) has shown that there exists a sequence u for which the equality holds in (13).

Since it is usually more convenient to use infinite sequences in practice, the result in (14) is most applicable. Therefore, for a given graph G one has $D^*(G,K,u) \geq T^*(G,K,u) \geq G'_{|P|}(\log K)^{|P|/2}/K$ where $G'_{|P|}$ is a function of $|P|$ only. Note the retardation on convergence due to $|P|$ and that this result holds for any sequence, random or nonrandom.

Finding a Sequence

Many sequences u are known to give faster convergence than pure random sampling. Of these at least two nonrandom sequences have the best known upper bound

$$D(N,K,u) \leq C_N(\log K)^N/K \ , \tag{15}$$

C_N being an increasing function of N and the particular sequence. Of these the one due to Halton (1960) is the most accessible. The other is due to Sobol (1967) and is described in Niederreiter (1978).

Let $p \geq 2$ be an integer. Then every non-negative integer n has an expansion of the form

$$n = \sum_{i=0}^{m} a_i p^i \qquad a_i \in \{0,1,\ldots,p-1\} \quad 0 \leq i \leq m \tag{16}$$

where $m = \lfloor \log_p n \rfloor$. Moreover, this p-adic representation is unique. Corresponding to (18), one has the radical inverse function

$$\phi_p(n) = \sum_{i=0}^{m} a_i p^{-i-1} \ . \tag{17}$$

The Halton sequence is $\{\phi_{p_1}(n+j),\ldots,\phi_{p_N}(n+j);\ j=1,\ldots,K\}$ where $p_1,\ldots,p_N$ are the first N prime numbers. These points are uniformly distributed in I_N . Actually, all that is needed to guarantee uniformity of distribution is that $p_1,\ldots,p_N$ be pairwise coprime. Note that for given n the sequence is not random. Since Halton also shows $T(N,K,u) \leq C'_N (\log K)^N/K$ for this sequence, the upper bounds on $D(N,K,u)$ and $T(N,K,u)$ apart from C_N and C'_N , are the same. Hereafter we concentrate on $D^*(G,K,u)$. In the present case $D^*(G,K,u) \leq (2^{|L|}-1)\, C_{|P|}(\log K)^{|P|}/K$. Halton and Smith (1964) describe an algorithm for generating the Halton sequence. By way of application Fishman (1983) shows the substantial benefit of using the Halton sequence to estimate the distribution functions of job completion time and shortest path time in a stochastic activity network with 18 arcs.

2. Using Cutsets

As just shown the rate of convergence of the upper bound on $D^*(G,K,u)$ depends on $(\log K)^{|P(s,t)|}$ where $|P(s,t)|$ is the dimension used for sampling. If one can reduce this dimension, then the bound on convergence can be improved. When they apply, series and parallel reductions and polygon-to-chain reductions (Satyanarayana and Wood 1982) induce lower dimensionality. This section describes an alternative method of reducing the dimensionality of sampling by using cutsets. By a cutset $R(s,t)$ we mean a set of edges that when inoperative disconnects s and t with probability 1. Here we need a more restricted cutset. Assume that no pathset connecting s and t contains more than one edge in the cutset. At least one such cutset is guaranteed to exist in G.

Let

$R(s,t)$ = a cutset of the network for s and t. (18a)

$L_r(s,t)$ = set of pathsets that use arc r $\quad r\epsilon R(s,t)$. (18b)

$R_r(s,t) = R(s,t) \cap P_r(s,t) \quad r\epsilon L(s,t)$. (18c)

Note that $R(s,t) = \cup_{r\epsilon L(s,t)} R_r(s,t)$ and that $L(s,t) = \sum_{r\epsilon R(s,t)} L_r(s,t)$.

Suppose that on replication j one chooses values only for $\underset{\sim}{u}_j(P-R)=\{u_{ij};\ i\epsilon(P-R)\}$. Then the conditional probability that s and t are connected is

$$g(s,t|\underset{\sim}{u}_j(P-R)) = 1-\Pi_{r\epsilon R}\{1-q_r[1-\Pi_{m\epsilon L_r}(1-B_j(P_m-R_m))]\}$$

$$= 1-\Pi_{r\epsilon R}[1-q_r \sum_{m=1}^{|L_r|}(-1)^m \tag{19}$$

$$\times \sum_{r_1<\ldots<r_m\epsilon L_r} B_j(\cup_{i=1}^{m}(P_{r_i}-R_{r_i}))] .$$

As a second estimator of g consider

$$\tilde{g}_K = \frac{1}{K} \sum_{j=1}^{K} g(s,t|\underset{\sim}{u}_j(P-R)) \ . \tag{20a}$$

After further expansion and rearrangement of terms in (19) one can write (20a) equivalently as

$$\tilde{g}_K = \frac{1}{K} \sum_{m=1}^{|L|}(-1)^{m+1} \sum_{r_1<\ldots<r_m \epsilon L} A(\cup_{i=1}^{m} W(P_{r_i}(s,t) - R_{r_i}(s,t))) \times \lambda(\cup_{i=1}^{m} W(R_{r_i}(s,t))) \ . \tag{20b}$$

For pure random sampling with independent replications, $\tilde{g}_K$ is unbiased and has $\text{var}\ \tilde{g}_K < \text{var}\ \hat{g}_K$.

For the more general setting for u the error bound on $\tilde{g}_K$ in terms of extreme discrepancy is

$$|\tilde{g}_K - g| \le D^{**}(G,K,u) = \sum_{m=1}^{|L|} \sum_{r_1<\ldots<r_m \epsilon L} \lambda(\cup_{i=1}^{m} W(R_{r_i})) \ D(|\cup_{i=1}^{m} (P_{r_i} - R_{r_i})|, K, u) \ . \tag{21}$$

This expression provides several useful insights. In particular, unlike the earlier sampling plan based on all $|P|$ arcs, the cutset approach enables one to use the structure of the network at hand to advantage. First, using the Halton or a like sequence gives

$$\begin{aligned} D^{**}(G,K,u) &\le \frac{1}{K} \sum_{m=1}^{|L|} \sum_{r_1<\ldots<r_m \epsilon L} \lambda(\cup_{i=1}^{m} W(R_{r_i})) \\ &\quad \times C_{|\cup_{i=1}^{m}(P_{r_i}-R_{r_i})|} (\log K)^{|\cup_{i=1}^{m}(P_{r_i}-R_{r_i})|} \\ &\le \frac{1}{K} (2^{|L|}-1) q_* C_{|P-R|} (\log K)^{|P-R|} \end{aligned} \tag{22}$$

where

$$q_* = \max_{i \in R} (q_i) .$$

Note that choosing R to be the cutset of maximal cardinality makes $(\log K)^{|P-R|}/K$ converge most rapidly. Moreover, (22) also suggests that among cutsets with maximal cardinality one should select the one for which $\lambda(W(R))$ is least. Note that $1-\lambda(W(R))$ is the probability that the cutset R fails. Although identifying cutsets with maximal cardinality is not a trivial problem, it is conceivable that one can select a cutset of sufficiently large cardinality to improve the convergence rate substantially. Sigal, Pritsker and Solberg (1980) propose the use of this cutset when Monte Carlo methods are used to estimate the distribution function of the shortest path time in a stochastic activity network. Also see Fishman (1983).

3. Other Reliability Measures

We now extend the analysis to other measures of reliability. Suppose one wants to estimate $g(s,T)$, the probability that vertex s is connected to all vertices $t \in T$. Observe that

$$\begin{aligned} g(s,T) &= 1-\text{pr}(\cup_{t \in T} (s \text{ and } t \text{ are not connected})) \\ &= 1- \sum_{m=1}^{|T|}(-1)^{m+1} \sum_{t_1<\ldots<t_m \in T} \text{pr}(\cap_{i=1}^{m} (s \text{ and } t_i \text{ are not connected})) \end{aligned} \tag{23}$$

Moreover, the conditional probability for replication j is

$$g(s,T|\underset{\sim}{u}_j) = \pi_{t \in T}[1-\pi_{r \in L(s,t)}(1-B_j(P_r(s,t))] . \tag{24}$$

As a first estimate of $g(s,T)$, one has

$$\hat{g}_K(s,T) = \frac{1}{K} \sum_{j=1}^{K} g(s,T|\underset{\sim}{u}_j) \tag{25}$$

Expansion by means of (6) shows that $|\hat{g}_K(s,T) - g(s,T)|$ has upper bound proportional to $(\log K)^N/K$, if the Halton or a like sequence is used, where

$$N = |U_{t\epsilon T} P(s,t)| \leq |M| \quad . \tag{26}$$

Note that for $T=V-\{s\}$ $g(s,T)$ denotes the probability that all nodes are connected.

The cutset approach to the estimation of $g(s,T)$ is somewhat more complex. Recall that $R(s,t)$ denotes a cutset with regard to vertices s and t. Now assume that the cutsets $\{R(s,t);\ t\epsilon T\}$ are identical. Denote this cutset by $R(s,T)$ and let $P(s,T) = U_{t\epsilon T}P(s,t)$. At least one such cutset must exist. Now the conditional probability that s is connected to all nodes in T on replication j is

$$g(s,T|\underset{\sim}{u}_j(P(s,T)-R(s,T))) = 1-\sum_{m=1}^{|T|}(-1)^{m+1} \sum_{t_1<\ldots<t_m\epsilon T} \tag{27}$$

$$\{\pi_{r\epsilon R(s,T)}\{1-q_r[1-\pi_{i=1}^{m} \pi_{k\epsilon L_r(s,t_i)}(1-B_j(P_k(s,t_i)-R_k(s,t_i)))]\}\} \quad .$$

Here the quantity in the outer braces is the conditional probability that s is connected to nodes $t_1,\ldots,t_m$. The justification for the form of (27) derives from the inclusion-exclusion expansion of (23). As a second estimate of $g(s,T)$, one has

$$\tilde{g}_K(s,T) = \frac{1}{K} \sum_{j=1}^{K} \tilde{g}(s,T|\underset{\sim}{u}_j(P(s,T) - R(s,T))) \tag{28}$$

for which $|\tilde{g}_K(s,T) - g(s,T)|$ has upper bound proportional to $(\log K)^N/K$ for the Halton sequence, where

$$N = |U_{t\epsilon T}P(s,t)| - |R(s,T)| \quad . \tag{29}$$

Now suppose that one wants to estimate $h(s,T)$, the mean number of s-t $(t\epsilon T)$ connections. Then one has

$$\hat{h}_K(s,T) = \sum_{t\epsilon T} \hat{g}_K(s,t) \tag{30a}$$

and

$$\tilde{h}_K(s,T) = \sum_{t\in T} \tilde{g}_K(s,t) \ . \tag{30b}$$

For $|\hat{h}_K(s,T) - h(s,T)|$ the exponent of $\log K$ in the upper bound is

$$N = \sup_{t\in T} |P(s,t)|$$

and for $|\tilde{h}_K(s,t) - h(s,T)|$ it is

$$N = \sup_{t\in T} |P(s,t) - R(s,t)| \ .$$

The estimation of at least two alternative conditional reliability measures also are of interest. First, suppose one wants to estimate $h(s,t|v)$, the probability that s and t are connected given that edge $v \in P(s,t)$ has failed. This measure characterizes the criticality of v to the s-t connection. Analogous to (9a) one has with the (s,t) argument suppressed

$$\hat{h}_K(s,t|v) = \frac{1}{K(1-q_v)} \sum_{j=1}^{K} I_{[q_v,1)}(u_{vj}) \{1-\pi_{r\in L} [1-B_j(P_r)]\} \ . \tag{31}$$

For the cutset approach with $v \notin R(s,t)$ one has

$$\tilde{h}_K(s,t|v) = \frac{1}{K(1-q_v)} \sum_{j=1}^{K} I_{[q_v,1)}(u_{vj}) \tag{32a}$$

$$\times \{1-\pi_{r\in R}[1-q_r(1-\pi_{m\in L_r}(1-B(P_m-R_m)))]\}$$

and for $v \in R(s,t)$

$$\tilde{h}_K(s,t|v) = \frac{1}{K} \sum_{j=1}^{K} \{1-\pi_{\substack{r\in R \\ r\neq v}} [1-q_r(1-\pi_{m\in L_r}(1-B(P_m-R_m)))]\} \ . \tag{32b}$$

The error bounds for $\hat{h}(s,t|v)$ and $\tilde{h}(s,t|v)$ remain the same as those for $\hat{g}(s,t)$ and $\tilde{g}(s,t)$ respectively.

The second conditional quantity of interest is $h(v|s,t)$, the probability that edge v fails ($v \in P(s,t)$), given that s and t do not communicate. Knowledge of this quantity can be helpful in detection problems. Here

$$\hat{h}_K(v|s,t) = \frac{[1-\hat{h}_K(s,t|v)](1-q_v)}{1-\hat{g}_K(s,t)} \tag{33}$$

and

$$\tilde{h}_K(v|s,t) = \frac{[1-\tilde{h}_K(s,t|v)](1-q_v)}{1-\tilde{g}_K(s,t)} \quad . \tag{34}$$

Expressions (33) and (34) call for more careful analysis than (32a) and (32b) do. Whereas in the case of pure random sampling $\hat{h}_K(s,t|v)$ and $\tilde{h}_K(s,t|v)$ are unbiased estimators, $\hat{h}_K(v|s,t)$ and $\tilde{h}_K(v|s,t)$ are biased estimators, at least for finite K. This is a consequence of both numerator and denominator in each being random variables. For the approach based on quasirandom points one may elect to assess errors for $|[1-\hat{h}_K(s,t|v)](1-q_v) - h(v|s,t)[1-\hat{g}_K(s,t)]|$ and $|[1-\tilde{h}_K(s,t|v)](1-q_v) - h(v|s,t)[1-\tilde{g}_K(s,t)]|$ which have $K^{-1}(\log K)^{|P(s,t)|}$ and $K^{-1}(\log K)^{|P(s,t)-R(s,t)|}$, respectively, in their upper bounds for the Halton and like sequences. Perhaps, it would be more direct to assess merely the errors for $[1-\hat{h}_K(s,t|v)](1-q_v)$ and $[1-\tilde{h}_K(s,t|v)](1-q_v)$ for all v of interest, since the denominators remain the same for all $v \in P(s,t)$.

4. Time Complexity Considerations

As mentioned earlier, the fact that computation of $g(s,t)$ in (1) has time complexity $O(2^{|M|})$ motivates one to consider methods of approximation; in particular, Monte Carlo methods. Therefore, it is only appropriate that we provide a comparative characterization of the time complexity of $\hat{g}_K(s,t)$ in (9a), $\hat{g}_K(s,T)$ in (25), $\hat{h}_K(s,T)$ in (30a), $\hat{h}_K(s,t|v)$ in (31), $\tilde{g}_K(s,t)$

in (20a), $\tilde{g}_K(s,T)$ in (28), $\tilde{h}_K(s,T)$ in (30b) and $\tilde{h}_K(s,t|v)$ in (32). Firstly and most importantly, although these cited expressions are convenient for error analysis, they are not intended for literal computation. We illustrate this point for $\hat{g}_K = \hat{g}_K(s,t)$.

Let

$$X_j = \Pi_{r\epsilon L(s,t)}[1-B_j(P_r(s,t))] \tag{35}$$

so that

$$\hat{g}_K = 1 - \frac{1}{K}\sum_{j=1}^{K} X_j \quad . \tag{36}$$

To compute X_j using (35) one needs to know $L(s,t)$ and $\{P_r(s,t);\ r\epsilon L(s,t)\}$, a burdensome requirement for a highly redundant network. Moreover, determining the status of each edge in $P(s,t)$ requires $|P(s,t)|$ steps and the straightforward computation of $B_j(P_r(s,t))$ requires $|P_r(s,t)|$ steps so that X_j has complexity $O(|P(s,t)|) + O(\sum_{r\epsilon L(s,t)}|P_r(s,t)|)$.

For this s-t connectedness problem a considerably more efficient approach exists for computing X_j . Basically if s and t are connected on replication j then $X_j=0$. Otherwise $X_j=1$. To check for connectedness in this single source-single sink setting one can use a relatively straightforward labeling algorithm which requires no more than $|V|$ steps. For example, see the discussion of Dykstra's algorithm in Aho, Hopcroft and Ullman (1974). As a result the time complexity for (36) is $O(K|P(s,t)|) + O(K|V|)$. Note that explicit knowledge of $L(s,t)$ and $\{P_r(s,t);\ r\epsilon L(s,t)\}$ is not required. Also note that this result applies to $\hat{h}_K(s,t|v)$ in (31) , as well for each v .

The (s,T) single source-multiple sink problem is slightly less routine. Again, labeling algorithms are known that can determine the connectedness of a single source s to all sinks t in T in no more

than $|V|$ steps. Using such an algorithm gives $\hat{g}_K(s,T)$ with time complexity $O(KN) + O(K|V|)$ where

$$N = |\cup_{t\in T} P(s,t)| \quad . \tag{37}$$

The same result holds for $\hat{h}_K(s,T)$.

When we turn to the cutset approach, the available results on complexity are mixed at present. Let $V(s,t)$ denote the set of vertices associated with the cutset $R(s,t)$ and consider the computation of $\tilde{g}_K(s,t)$. Let

$$X_{rj} = \Pi_{m\in L_r(s,t)} [1-B_j(P_m(s,t)-R_m(s,t))] \quad r\in R(s,t) \tag{38}$$

so that (20a) is algebraically equivalent to

$$\tilde{g}_K = \frac{1}{K} \sum_{j=1}^{K} \Pi_{r\in R(s,t)} [1-q_r X_{rj}] \; . \tag{39}$$

Now there is a labeling algorithm for which the time complexity of determining the s-v connections for all $v \in V(s,t)$ is $O(|V|)$ and also the time complexity of determining the t-v connections for all $v \in V(s,t)$ is $O(|V|)$. Since the status of the s-v and v-t connections is all that is necessary for evaluating X_{rj} in (38) this computation has time complexity $O(|P(s,t) - R(s,t)|) + O(|V|)$. Therefore, $\tilde{g}_K$ has time complexity $O(K|P(s,t) - R(s,t)|) + O(K|V|)$. A little thought shows a similar analysis for the elements of $\tilde{h}_K(s,T)$. However, no comparable complexity has been established for $\tilde{g}_K(s,T)$. This remains a topic for continued research.

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1. REPORT NUMBER UNC/ORSA/TR-83/2	2. GOVT ACCESSION NO. AD-A131 849	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) AN ALTERNATIVE TO THE MONTE CARLO ESTIMATION OF NETWORK RELIABILITY		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) GEORGE S. FISHMAN		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0302
9. PERFORMING ORGANIZATION NAME AND ADDRESS CURRIC. IN OPERATIONS RESEARCH & SYSTEMS ANALYSIS SMITH BUILDING 128A UNIV. OF NORTH CAROLINA CHAPEL HILL, NC 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS NAVAL ANALYSIS PROGRAM OFFICE OF NAVAL RESEARCH ARLINGTON, VA 22217		12. REPORT DATE AUGUST 1983 13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

MONTE CARLO METHODS, NETWORK RELIABILITY, QUASIRANDOM POINTS, VARIANCE REDUCTION

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper describes a numerical method based on quasirandom points for estimating the probability g(s,t) that vertices s and t are connected in an undirected network G=(V,M) with |V| perfect vertices and |M| randomly and independently failing edges. It is shown that the error

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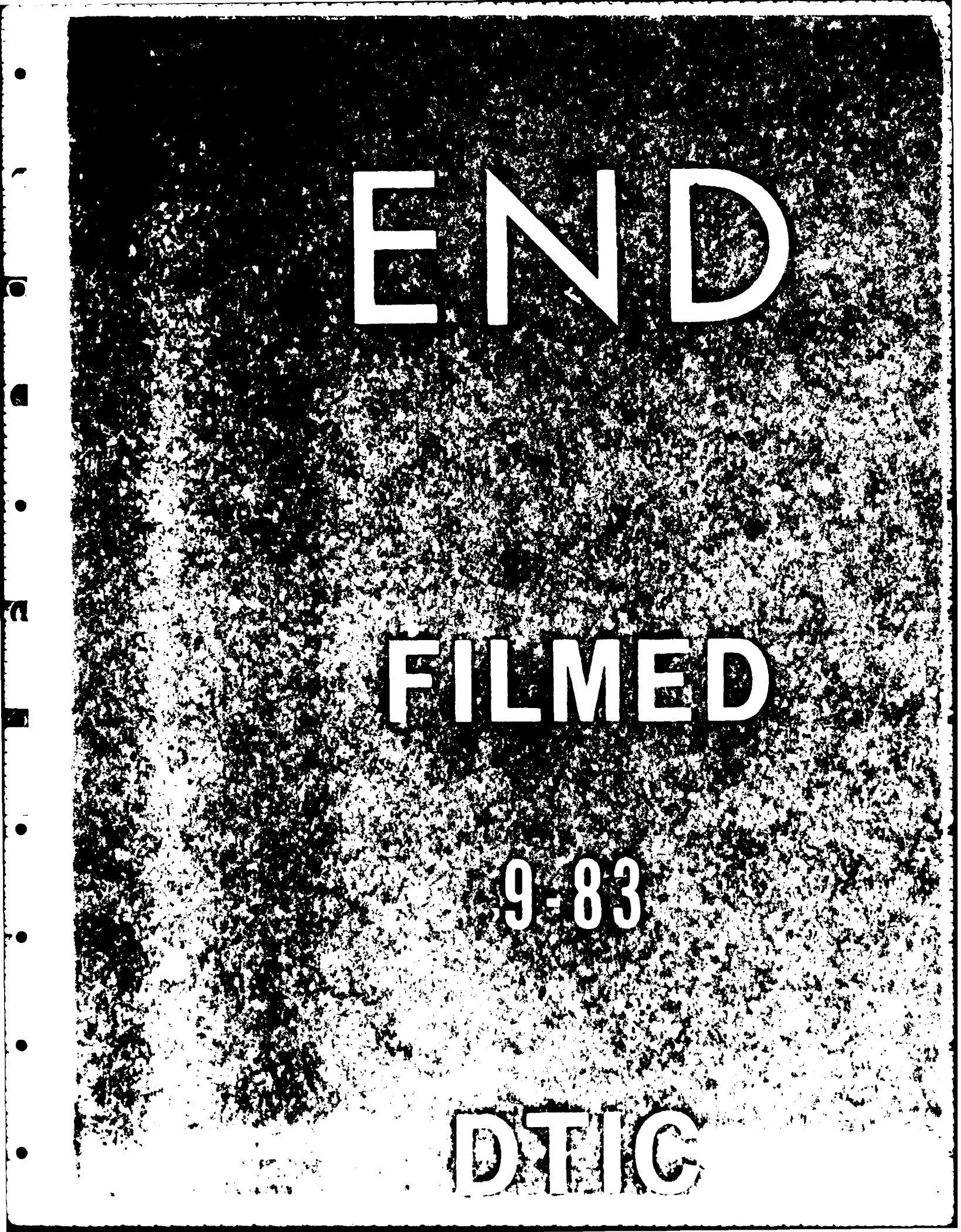
of estimate, as measured in terms of extreme discrepancy, has a lower bound proportional to $(\log K)^{|P(s,t)|/2}/K$ for all possible sampling sequences, but has an upper bound proportional to $(\log K)^{|P(s,t)|}/K$ for certain quasirandom sequences, where $P(s,t)$ M is the set of edges on the paths that connect s and t and K is the number of replications. By comparison previously proposed sampling methods for this problem all lead to a standard error of estimate proportional to $K^{-1/2}$. Moreover, since quasirandom points are not random, the associated bounds are deterministic. By using a cutset $R(s,t)$ with a certain special property these lower and upper bounds become, respectively, $(\log K)^{|P(s,t)-R(s,t)|/2}/K$ and $(\log K)^{|P(s,t)-R(s,t)|}/K$. This suggests that one choose a cutset $R(s,t)$ with maximal cardinality. Also, it is shown that the coefficient of $(\log K)^{|P(s,t)-R(s,t)|}/K$ is least if among all cutsets with maximal cardinality one chooses the cutset with the largest failure probability. The paper extends the results to more global measures of reliability and to conditional measures of reliability. A discussion of the computation time complexity of the proposed method is also included.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